Non-separable UHF algebras (Problems in theory of operator algebras)

Katsura, Takeshi

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Takeshi Katsura (勝良 健史)
Department of Mathematics, Keio University,
3–14–1 Hiyoshi, Yokohama 223–8522, JAPAN
e-mail: katsura@math.keio.ac.jp

0 Introduction

In this short note, we review results in [FK1, FK2] of a joint work with Ilijas Farah. In [FK1], we completely solved an old question of Dixmier on definition of UHF algebras. In [FK2], we gave a non-classification results of non-separable UHF type C*-algebras.

1 Three definitions of UHF algebras

In this section, we introduce three concepts which we name UHF, AM and LM. These three notions coincide for separable unital C*-algebras, and Dixmier asked whether these are still equivalent for non-separable unital C*-algebras. We prove that this is not the case for most cases.

For two subsets $\mathcal{F}, \mathcal{G}$ of a C*-algebra A and $\varepsilon > 0$, we write $\mathcal{F} \subset_\varepsilon \mathcal{G}$ if for all $x \in \mathcal{F}$ there exists $y \in \mathcal{G}$ such that $\|x - y\| < \varepsilon$. For each positive integer n, we denote by $M_n(\mathbb{C})$ the unital C*-algebra of all $n \times n$ matrices with complex entries. A C*-algebra which is isomorphic to $M_n(\mathbb{C})$ for some n is called a *full matrix algebra.*

Definition 1.1 A C*-algebra A is said to be

- *uniformly hyperfinite* (or *UHF*) if A is isomorphic to a tensor product of full matrix algebras.
- *approximately matricial* (or *AM*) if it has a directed family of full matrix subalgebras with dense union.
- *locally matricial* (or *LM*) if for any finite subset $\mathcal{F}$ of A and any $\varepsilon > 0$, there exists a full matrix subalgebra M of A with $\mathcal{F} \subset_\varepsilon M$,

In [G, Theorem 1.13], Glimm shows that for separable C*-algebras, the three conditions UHF, unital AM and unital LM coincide (see also [D, Remark 1.3 and Theorem 1.6]). Dixmier asked whether these three conditions coincide for general C*-algebras in [D, Problem 8.1].

By definition, AM algebra is an inductive limit of full matrix algebras. It is easy to see, using the result on separable UHF algebras mentioned above, that LM algebra is an inductive limit of separable UHF algebras. Thus the existence of an LM algebra which is not AM, shown in Theorem 3.1, implies that the class of inductive limits of full matrix algebras is not closed under taking inductive limits.

2 Cardinals

We prepare some notions on cardinals in this section before stating the main results in the next section. Consult [J] for detail on cardinals and their arithmetic.

For an ordinal γ, $\aleph_\gamma$ is the γ-th infinite cardinal (with counting starting at $\aleph_0$ as the 0-th infinite cardinal). Hence $\aleph_0$ is the smallest infinite cardinal, and $\aleph_1$ is the smallest uncountable cardinal and so on. Let us denote the cardinality of a set X by $|X|$. Thus $|\mathbb{N}| = \aleph_0$ and $|\mathbb{R}| = 2^{\aleph_0}$. Our results do not use any axiom other than the standard axioms ZFC, but some problems relate to the Continuum Hypothesis. Recall that the Continuum Hypothesis asks whether $2^{\aleph_0} = \aleph_1$ holds or not, and is known to be independent of ZFC.

Definition 2.1 An infinite cardinal κ is said to be *singular* if there exists a set I with $|I| < \kappa$ and cardinals κ_i with $\kappa_i < \kappa$ for $i \in I$ such that $\kappa = \sum_{i\in I} \kappa_i$. Otherwise κ is said to be *regular.*

The cardinals $\aleph_1, \aleph_2, \ldots$ are regular. More generally, every successor cardinal $\aleph_{\gamma+1}$ is regular because $\aleph_\gamma \aleph_\gamma = \aleph_\gamma$. Let ω be the smallest infinite ordinal. Then the limit cardinal $\aleph_\omega$ is singular because $\aleph_\omega = \sum_{n\in\mathbb{N}} \aleph_n$.

Definition 2.2 The *character density* $\chi(A)$ of a C*-algebra A is the smallest cardinality of a dense subset of A.

Note that A is separable if and only if its character density $\chi(A)$ is $\aleph_0$.

Lemma 2.3 *For an infinite cardinal κ, there are at most 2^κ isomorphism classes of C*-algebras A with $\chi(A) = \kappa$.*

Proof. This is because on a fixed set of size κ there are at most 2^κ ways to define $+, \cdot, {}^*$ and $\|\cdot\|$. □

3 Main results

The following is the main result in [FK1].

Theorem 3.1 (1) *For a C*-algebra with character density at most $\aleph_1$, AM and LM are equivalent.*

(2) *For every cardinal $\kappa > \aleph_1$, there exists a unital LM algebra with character density κ which is not AM.*

(3) *For every cardinal $\kappa \geq \aleph_1$, there exists a unital AM algebra with character density κ which is not UHF.*

Similarly, (1) and (2) hold if we replace AM and LM to AF and LF defined as follows.

Definition 3.2 A C*-algebra A is said to be

- *approximately finite-dimensional* (or *AF*) if it has a directed family of finite-dimensional subalgebras with dense union.
- *locally finite-dimensional* (or *LF*) if for any finite subset $\mathcal{F}$ of A and any $\varepsilon > 0$, there exists a finite-dimensional subalgebra D of A with $\mathcal{F} \subset_\varepsilon D$,

In the next section, we give classification of UHF algebras done in [FK2] (Proposition 4.1). From this, we get the following.

Proposition 3.3 *Let γ be an ordinal. Then the number of isomorphism classes of UHF algebras of character density $\aleph_\gamma$ is equal to $(|\gamma| + \aleph_0)^{\aleph_0}$.*

By setting $\gamma = 0$ in the proposition above, we see that the number of isomorphism classes of separable UHF algebras is $\aleph_0^{\aleph_0} = 2^{\aleph_0}$. Hence there are as many separable UHF algebras as separable C*-algebras. However for $\gamma > 0$, the cardinality $(|\gamma| + \aleph_0)^{\aleph_0}$ is much, much smaller than the (possible) cardinality $2^{\aleph_\gamma}$ of the isomorphism classes of C*-algebras of character density $\aleph_\gamma$ (see Lemma 2.3). For example, $(|\gamma| + \aleph_0)^{\aleph_0}$ is still $2^{\aleph_0}$ for an ordinal γ with $|\gamma| \leq 2^{\aleph_0}$. On the contrary, the following theorem, which is one of the main result in [FK2], says that there are lots of AM algebras.

Theorem 3.4 *If κ is a regular infinite cardinal, then there are 2^κ non-isomorphic AM algebras of character density κ.*

In fact, all AM algebras in Theorem 3.4 can be chosen so that they have the same K-groups and Cuntz semigroups as the ones of the CAR algebra. For a singular infinite cardinal κ we do not know whether there are 2^κ non-isomorphic AM algebras of character density κ. For example, when $\kappa = \aleph_\omega$, we know that there are $\sum_{n\in\mathbb{N}} 2^{\aleph_n}$ non-isomorphic AM algebras of character density $\aleph_\omega$ using Theorem 3.4. However this cardinality can be strictly smaller than $2^{\aleph_\omega} = \prod_{n\in\mathbb{N}} 2^{\aleph_n}$.

4 Classification of UHF algebras

Separable UHF algebras was classified by supernatural numbers. In this section, we extend this classification to general UHF algebras using what I call hypernatural numbers. In this note, a hypernatural number λ means that a family of cardinals $\lambda = (\lambda_p)_{p\in\mathcal{P}}$ indexed by the set $\mathcal{P}$ of all prime numbers. A hypernatural number λ can be symbolically written as $\lambda = \prod_{p\in\mathcal{P}} p^{\lambda_p}$. For a hypernatural number $\lambda =$

$(\lambda_p)_{p\in\mathcal{P}}$, the cardinal $\sum_{p\in\mathcal{P}} \lambda_p$ is called the density of λ. Recall that a supernatural number can be considered as a hypernatural number of density $\leq \aleph_0$.

A UHF algebra is a C*-algebra obtained as a tensor product $A = \bigotimes_{x\in X} M_{n(x)}(\mathbb{C})$ where $n\colon X \to \mathbb{Z}_{\geq 2}$ is a map from a set X to the set $\mathbb{Z}_{\geq 2}$ of the integers greater than or equal to 2. One can see that $\chi(A) = |X|$. For such a map $n\colon X \to \mathbb{Z}_{\geq 2}$ we associate a hypernatural number λ of density $|X|$ by

$$\lambda_p := \sum_{x\in X} \max\{l \in \mathbb{N} : p^l \mid n(x)\}$$

for each prime $p \in \mathcal{P}$. We say that λ is the hypernatural number of the UHF algebra $A = \bigotimes_{x\in X} M_{n(x)}(\mathbb{C})$. We prove the following in [FK2].

Proposition 4.1 *The correspondence described above from a UHF algebra of character density κ to its hypernatural number of density κ is one-to-one.*

For a cardinal κ, we denote by $[0,\kappa]$ the set of all cardinal less than or equal to κ. Note that we have $|[0,\aleph_\gamma]| = |\gamma| + \aleph_0$ for an ordinal γ. In other words, when $\kappa = \aleph_0, \aleph_1, \ldots, \aleph_n, \ldots$ we have $|[0,\kappa]| = \aleph_0$, and when $\kappa = \aleph_\gamma$ for an infinite ordinal γ, we have $|[0,\kappa]| = |\gamma|$. From Proposition 4.1, the set of isomorphism classes of UHF algebras of character density $\leq \aleph_\gamma$ corresponds bijectively to the set of maps from $\mathcal{P}$ to $[0,\aleph_\gamma]$. This proves Proposition 3.3.

I have a complaint about the discussion above on hypernatural numbers (other than the name "hypernatural number"). That is the above definition of the hypernatural number of a UHF algebra depends on particular tensor products representations of a given UHF algebra. In the proof of Proposition 4.1, we show that this notion actually does not depend on tensor products representations, but it is better to give a definition without using tensor products representations. There is an obvious candidate, but I do not know this definition coincides with the one above. Specifically the following is open.

Conjecture 4.2 *Let A be a UHF algebra, and $\lambda = \prod_{p\in\mathcal{P}} p^{\lambda_p}$ be its hypernatural number described above. Then for each $p \in \mathcal{P}$ and a cardinal κ there exists a unital $*$-homomorphism from $\bigotimes_\kappa M_p(\mathbb{C})$ to A if and only if $\kappa \leq \lambda(p)$.*

If this conjecture is true, then one can define λ_p for A to be the supremum (which is actually maximum) of cardinalities κ such that there exists a unital $*$-homomorphism from $\bigotimes_\kappa M_p(\mathbb{C})$ to A. This definition is intrinsic, and does not use tensor products representations of a UHF algebra A. Of course, the "if" part can be proven easily. It is also easy to see the "only if" part for $\kappa \leq \aleph_0$. I do not know how to prove the following.

Problem 4.3 *Prove that there exists no unital $*$-homomorphism from $\bigotimes_{\aleph_1} M_3(\mathbb{C})$ to $\left(\bigotimes_{\aleph_1} M_2(\mathbb{C})\right) \otimes \left(\bigotimes_{\aleph_0} M_3(\mathbb{C})\right)$.*

5 Twisted action and Twisted crossed products

To prove Theorem 3.1 and Theorem 3.4, we use crossed products or more generally twisted crossed products. In this section we give definitions and examples of twisted action and twisted crossed products. For more detail, see [PR].

Let A be a unital C*-algebra. The automorphism group and the unitary group of A are denoted by $\mathrm{Aut}(A)$ and $U(A)$, respectively. The units of $\mathrm{Aut}(A)$ and $U(A)$ are denoted by id and 1 although the units of all other groups are denoted by e. For a unitary $u \in U(A)$, we define $\mathrm{Ad}\, u \in \mathrm{Aut}(A)$ by $\mathrm{Ad}\, u(a) = uau^*$ for $a \in A$.

Definition 5.1 A *twisted action* of a group G on a unital C*-algebra A is a pair (α, c) of maps $\alpha\colon G \to \mathrm{Aut}(A)$ and $c\colon G \times G \to U(A)$ satisfying

(i) $\alpha_e = \mathrm{id}$ and $\alpha_g\alpha_h = \mathrm{Ad}(c(g,h))\alpha_{gh}$ for all $g, h \in G$,

(ii) $c(e,g) = c(g,e) = 1$ for all $g \in G$ and
$c(g,h)c(gh,k) = \alpha_g(c(h,k))c(g,hk)$ for all $g,h,k \in G$.

If $c(g,h) = 1$ for all $g, h \in G$, then $\alpha\colon G \to \mathrm{Aut}(A)$ is a homomorphism. In this case we say that α is an action.

Definition 5.2 For a twisted action (α, c) of a group G on a unital C*-algebra A, we define its (full) *twisted crossed product* $A \rtimes_{\alpha,c} G$ to be the universal unital C*-algebra generated by a unital subalgebra $A \subset A \rtimes_{\alpha,c} G$ and a family $\{u_g\}_{g\in G}$ of unitaries in $A \rtimes_{\alpha,c} G$ such that $u_g u_h = c(g,h)u_{gh}$ for $g, h \in G$ and $\mathrm{Ad}\, u_g|_A = \alpha_g$ for $g \in G$.

For $g \in G$, the unitary $u_g \in A \rtimes_{\alpha,c} G$ is called the *implementing unitary* of α_g. It is easy to see that the linear span of the elements in the form $au_g \in A \rtimes_{\alpha,c} G$ for $a \in A$ and $g \in G$ is dense in $A \rtimes_{\alpha,c} G$.

For our purpose, we only need the existence of a unital C*-algebra generated by a unital subalgebra A and a family $\{u_g\}_{g\in G}$ of unitaries satisfying the two conditions in Definition 5.2 for the twisted actions (α, c) considered below, and we do not use its universal property. In fact, since every our example is simple any such a C*-algebra is universal. We can use a regular representation to prove the existence of such a C*-algebra, and we do in [FK1]. Although in [PR] it was assumed that a C*-algebra is separable and a group is second countable, these assumptions are needed only to deal with Borel maps. Because we consider discrete groups, all results in [PR] hold for non-separable C*-algebras and uncountable groups (cf. [ZM]).

Let us consider a twisted action (α, c) of a group G on the trivial unital C*-algebra $A = \mathbb{C}$. Since $\mathrm{Aut}(\mathbb{C})$ consists of the one element $\{\mathrm{id}\}$, there exists only one choice for the map $\alpha\colon G \to \mathrm{Aut}(\mathbb{C})$, namely the trivial one. In this case, the twisted crossed product $\mathbb{C} \rtimes_{\alpha,c} G$ is called a *twisted group C*-algebra*, and denoted by $C^*(G; c)$. The map $c\colon G \times G \to U(\mathbb{C}) =: \mathbb{T}$ satisfying the condition (ii) in Definition 5.1 for trivial α is called a *2-cocycle*. All groups considered in this paper are of the following type.

Definition 5.3 *For a set X, we denote by G_X the abelian group consisting of all finite subsets of X where the operation is the symmetric difference Δ.*

Note that G_X is isomorphic to the direct sum $\bigoplus_X \mathbb{Z}/2\mathbb{Z}$ of $|X|$-copies of $\mathbb{Z}/2\mathbb{Z}$. We often identify an element x of X with a subset $\{x\}$ of X. For these group G_X, examples of 2-cocycles c and twisted group C*-algebras $C^*(G_X; c)$ are given by the following lemma. For a set X, we denote by $[X]^2$ the set of all subsets of X with cardinality 2.

Lemma 5.4 *Let X be a set. Take $Z \subset X^2$ such that $(x, y) \in Z$ implies $(y, x) \notin Z$, and set $[Z] = \{\{x, y\} \in [X]^2 : (x, y) \in Z\} \subset [X]^2$. Then the map $c_Z\colon G_X \times G_X \to \mathbb{T}$ defined by*

$$c_Z(g, h) := (-1)^{|\{(x,y)\in Z: x\in g, y\in h\}|}.$$

is a 2-cocycle, and the twisted group C^*-algebra $C^*(G_X; c_Z)$ is the universal C^*-algebra generated by self-adjoint unitaries $\{u_x\}_{x\in X}$ satisfying that $u_x u_y = u_y u_x$ for $\{x, y\} \notin [Z]$ and $u_x u_y = -u_y u_x$ for $\{x, y\} \in [Z]$.

Proof. It is straightforward to check that c_Z is a 2-cocycle. Let $\{u_g\}_{g\in G_X}$ be the generators in $C^*(G_X; c_Z)$ given in Definition 5.2. Then it is easy to see that the family $\{u_{\{x\}}\}_{x\in X}$ satisfies the conditions in the statement. Conversely, take self-adjoint unitaries $\{u_x\}_{x\in X}$ satisfying the conditions in the statement. For $g = \{x_1, x_2, \ldots, x_n\} \in G_X$, set u_g by

$$u_g := (-1)^k u_{x_1} u_{x_2} \cdots u_{x_n}$$

where $k := |\{(i, j) : 1 \le i < j \le n,\ (x_i, x_j) \in Z\}|$. Then it is routine to check that this is well-defined and $\{u_g\}_{g\in G_X}$ satisfies the conditions for the generators in $C^*(G_X; c_Z)$. This concludes the proof. □

Remark 5.5 We note that $C^*(G_X; c_Z)$ depends only on $[Z] \subset [X]^2$, and every subset of $[X]^2$ arises. It is known that the twisted group C*-algebra $C^*(G; c)$ only depends on the class defined by a 2-cocycle c in the 2-cohomology group $H^2(G; \mathbb{T})$ (see cite[Lemma 3.3]PR for example). When the group G is given by $G = G_X$ for a set X, the 2-cohomology group $H^2(G_X; \mathbb{T})$ is isomorphic to the group $2^{[X]^2}$ of all subsets of $[X]^2$ (which is isomorphic to $\prod_{[X]^2} \mathbb{Z}/2\mathbb{Z}$) via the map $c_Z \mapsto [X]^2 \setminus [Z]$. Thus the above lemma gives a description of all twisted group C*-algebras $C^*(G; c)$ when $G = G_X$ for a set X. We do not use these facts.

If $|X| = 2$, say $X = \{+, -\}$, and if $Z = \{(+, -)\} \subset X^2$, then the twisted group C*-algebra $C^*(G_X; c_Z)$ considered in Lemma 5.4 is the C*-algebra generated by two self-adjoint unitaries u_+, u_- with $u_+ u_- = -u_- u_+$. This C*-algebra is 4-dimensional, its basis is given by $\{1, u_+, u_-, u_{\{+,-\}}\}$, and it is noncommutative. Hence it is isomorphic to $M_2(\mathbb{C})$. The concrete isomorphism from $C^*(G_X; c_Z)$ to $M_2(\mathbb{C})$ is given by sending u_+ and u_- to the unitaries

$$v = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{and} \qquad w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

in $M_2(\mathbb{C})$. In summary, we get the following.

Lemma 5.6 *A C^*-algebra $C^*(\{v,w\})$ generated by two self-adjoint unitaries v, w with $vw = -wv$ is always isomorphic to $M_2(\mathbb{C})$.*

Let us take a set X and fix it. We consider the set $Y := X \times \{+,-\}$ and

$$Z := \big\{((x,+),(x,-)) : x \in X\big\} \subset Y^2.$$

We denote by A_X the twisted group C*-algebra $C^*(G_Y; c_Z)$. By Lemma 5.4, A_X is the C*-algebra generated by self-adjoint unitaries $\{u_{(x,+)}, u_{(x,-)}\}_{x\in X}$ with $u_{(x,+)}u_{(x,-)} = -u_{(x,-)}u_{(x,+)}$ for $x \in X$ and $u_{(x,i)}u_{(y,j)} = u_{(y,j)}u_{(x,i)}$ for $x \neq y$ and $i,j \in \{+,-\}$. By Lemma 5.6, one can see that $C^*(\{u_{(x,+)}, u_{(x,-)}\}) \cong M_2(\mathbb{C})$ for each $x \in X$. Since the family $\big\{C^*(\{u_{(x,+)}, u_{(x,-)}\})\big\}_{x\in X}$ mutually commutes and generates $A_X = C^*(G_Y; c_Z)$, we get the following.

Lemma 5.7 *The twisted group C^*-algebra $A_X = C^*(G_Y; c_Z)$ is isomorphic to the UHF algebra $\bigotimes_{x\in X} M_2(\mathbb{C})$ of character density $|X|$.*

When $|X| = \aleph_0$, the UHF algebra $\bigotimes_{x\in X} M_2(\mathbb{C})$ is called the CAR algebra. Let $\alpha \in \mathrm{Aut}(A_X)$ be the automorphism such that $\alpha|_{C^*(\{u_{(x,+)},u_{(x,-)}\})} = \mathrm{Ad}(u_{(x,+)})$ for $x \in X$. Since $\alpha^2 = \mathrm{id}$, we can consider α as an action of $\mathbb{Z}/2\mathbb{Z}$ on A_X. We define B_X to be the crossed product $A_X \rtimes_\alpha (\mathbb{Z}/2\mathbb{Z})$. Note that for a subset F of X, we can consider A_F and B_F as subalgebras of A_X and B_X, respectively. We are going to see that B_X can be seen as a twisted group C*-algebra.

We set

$$Y_\bullet := Y \cup \{\bullet\} = (X \times \{+,-\}) \cup \{\bullet\},$$

and

$$Z_\bullet := \{((x,+),(x,-)) : x \in X\} \cup \{(\bullet,(x,-)) : x \in X\} \subset Y_\bullet^2.$$

Then the twisted group C*-algebra $C^*(G_{Y_\bullet}; c_{Z_\bullet})$ is generated by $C^*(G_Y; c_Z) \subset C^*(G_{Y_\bullet}; c_{Z_\bullet})$ and the self-adjoint unitary $u_\bullet \in C^*(G_{Y_\bullet}; c_{Z_\bullet})$. For each $x \in X$, we have

$$\mathrm{Ad}\, u_\bullet|_{C^*(\{u_{(x,+)},u_{(x,-)}\})} = \mathrm{Ad}\, u_{(x,+)}|_{C^*(\{u_{(x,+)},u_{(x,-)}\})} = \alpha|_{C^*(\{u_{(x,+)},u_{(x,-)}\})}$$

Hence the universalities show the following.

Lemma 5.8 *There is an isomorphism from the crossed product $B_X = A_X \rtimes_\alpha (\mathbb{Z}/2\mathbb{Z})$ to the twisted group C^*-algebra $C^*(G_{Y_\bullet}; c_{Z_\bullet})$ preserving $A_X = C^*(G_Y; c_Z)$ and sending the implementing unitary of α in B_X to $u_\bullet \in C^*(G_{Y_\bullet}; c_{Z_\bullet})$.*

In [FK1], we prove the following.

Theorem 5.9 *For an infinite X, the crossed product $B_X = A_X \rtimes_\alpha (\mathbb{Z}/2\mathbb{Z})$ is a unital AM algebra of character density $|X|$. It is UHF if and only if $|X| = \aleph_0$.*

This shows (3) of Theorem 3.1. We sketch its proof in Section 7. In this section, we give an idea of the proof of Theorem 3.1 using twisted group C*-algebras.

Proposition 5.10 *For an infinite set X, there is an isomorphism $\iota\colon G_Y \to G_{Y_\bullet}$ such that $c_Z = c_{Z_\bullet} \circ (\iota \times \iota)$ if and only if $|X| = \aleph_0$.*

Proof. Suppose $|X| = \aleph_0$. Then one can identify X and $\mathbb{N}$. Let $\iota\colon G_Y \to G_{Y_\bullet}$ be the group homomorphism such that $\iota(\{(0,+)\}) = \{\bullet\}$ and

$$\iota(\{(n,+)\}) = \{\bullet, (0,+), (1,+), \ldots, (n-1,+)\} \in G_{Y_\bullet}$$
$$\iota(\{(n,-)\}) = \{(n-1,-), (n,-)\} \in G_{Y_\bullet}$$

for $n \in \mathbb{N} = X$. Then it is routine to check that ι is an isomorphism satisfying $c_Z = c_{Z_\bullet} \circ (\iota \times \iota)$.

Next suppose $|X| > \aleph_0$ and assume that there is an isomorphism $\iota\colon G_Y \to G_{Y_\bullet}$ such that $c_Z = c_{Z_\bullet} \circ (\iota \times \iota)$. We are going to derive a contradiction. Take a finite subset $F_1 \subset X$ such that $\{\bullet\} \in \iota(G_{F_1 \times \{+,-\}})$. Next take a finite subset $F_1' \subset X$ such that $\iota(G_{F_1 \times \{+,-\}}) \subset G_{(F_1' \times \{+,-\}) \cup \{\bullet\}}$. Then take a finite subset $F_2 \subset X$ containing F_1 and satisfying $G_{(F_1' \times \{+,-\}) \cup \{\bullet\}} \subset \iota(G_{F_2 \times \{+,-\}})$. Inductively, we will find two increasing sequences $\{F_n\}_{n=1}^\infty$ and $\{F_n'\}_{n=1}^\infty$ of finite subsets of X such that

$$\iota(G_{F_n \times \{+,-\}}) \subset G_{(F_n' \times \{+,-\}) \cup \{\bullet\}} \subset \iota(G_{F_{n+1} \times \{+,-\}})$$

holds for all n. Let us define countable sets $F, F' \subset X$ by $F = \bigcup_{n=1}^\infty F_n$ and $F' = \bigcup_{n=1}^\infty F_n'$. Then we have $\iota(G_{F \times \{+,-\}}) = G_{(F' \times \{+,-\}) \cup \{\bullet\}}$. We have

$$\{g \in G_Y : c_Z(g,f) = 1 \text{ for all } f \in G_{F \times \{+,-\}}\} = G_{(X \setminus F) \times \{+,-\}}$$

and $G_Y = G_{F \times \{+,-\}} \times G_{(X \setminus F) \times \{+,-\}}$. Since ι satisfies $c_Z = c_{Z_\bullet} \circ (\iota \times \iota)$, the subgroup $G_{(F' \times \{+,-\}) \cup \{\bullet\}} \subset G_{Y_\bullet}$ should satisfy the same property. Namely we should have $G_{Y_\bullet} = G_{(F' \times \{+,-\}) \cup \{\bullet\}} \times G'$ where

$$G' := \{g \in G_{Y_\bullet} : c_{Z_\bullet}(g,f) = 1 \text{ for all } f \in G_{(F' \times \{+,-\}) \cup \{\bullet\}}\}.$$

It is not difficult to see that $g \in G'$ if and only if

$$g = \{(x,+) : x \in A\} \cup \{(x,-) : x \in B\}$$

for finite subsets $A, B \subset X \setminus F'$ such that the cardinality of B is even. Since $|X| > \aleph_0$, one can find $x \in X \setminus F'$. Then $\{(x,-)\} \in G_{Y_\bullet}$ is not in $G_{(F' \times \{+,-\}) \cup \{\bullet\}} \times G'$. This is a contradiction. □

This proposition explains the second statement of Theorem 5.9. In fact, this gives the proof for the "if" part, and one can show the "only if" part in a similar way to the proof of Proposition 5.10 (see Section 7). It is not difficult to see the first statement of Theorem 5.9, but we just see that B_X is LM below. We use the proof of this proposition in Section 7.

Proposition 5.11 *For an infinite set X, the twisted group C^*-algebra $C^*(G_{Y_\bullet}; c_{Z_\bullet}) \cong B_X$ is an inductive limit of CAR algebras, and hence LM.*

Proof. Let $[X]^{\aleph_0}$ be the directed set of all subsets $Y \subset X$ with $|Y| = \aleph_0$ directed by the inclusion. We see that $\{B_Y\}_{Y \in [X]^{\aleph_0}}$ is a directed family of B_X with dense union. By Proposition 5.10, B_Y is isomorphic to A_Y which is the CAR algebra for all $Y \in [X]^{\aleph_0}$. We are done. □

6 LM algebra which is not AM

In the previous section, we see how the difference between $\aleph_0$ and the cardinal greater than $\aleph_0$ affects the property UHF. In this section, we try to explain how the difference between $\aleph_1$ and the cardinal greater than $\aleph_1$ affects the property AM.

In the proof for the fat that a separable LM algebra is AM, the point is to use a sequence rather than a general directed set. This argument can be generalised to show that an LM algebra with character density at most $\aleph_1$ is AM ((1) of Theorem 3.1) because of the following lemma.

Lemma 6.1 *A set X can be written as a union of totally ordered (by the inclusion) family of countable subsets if and only if $|X| \leq \aleph_1$.*

Proof. If $|X| = \aleph_1$, then we can identify X with the smallest uncountable ordinal ω_1. For each $x \in \omega_1$, we define $C_x = \{y \in \omega_1 : y < x\}$. Then $\{C_x\}_{x\in\omega_1}$ is a totally ordered (even well ordered) family of countable subsets whose union is whole ω_1.

Let X be a set with $|X| > \aleph_1$, and suppose that X can be written as a union of totally ordered family $\{C_\lambda\}_{\lambda\in\Lambda}$ of countable subsets of X. We are going to derive a contradiction. Take $Z \subset X$ with $|Z| = \aleph_1$. For each $x \in Z$, choose $\lambda_x \in \Lambda$ such that $x \in C_{\lambda_x}$. Since the cardinality of $\bigcup_{x\in Z} C_{\lambda_x} \subset X$ is $\aleph_1$, we can choose $y \in X$ such that $y \notin C_{\lambda_x}$ for all $x \in Z$. Take $\lambda \in \Lambda$ with $y \in C_\lambda$. Since C_λ is countable, one can choose $z \in Z$ with $z \notin C_\lambda$. Then we have neither $C_{\lambda_z} \subset C_\lambda$ nor $C_\lambda \subset C_{\lambda_z}$. This is a contradiction. □

Combining this lemma with several arguments, one can show (1) and (2) of Theorem 3.1, In the rest of this section, we give the construction of an LM algebra which is not AM using twisted crossed product.

Let X be a set, and $[X]^2$ be the set of all subsets of X with cardinality 2. For $\xi = \{x, y\} \in [X]^2$ let A_ξ be the CAR algebra. We fix four self-adjoint unitaries $v_{x,y}, v_{y,x}, w_{x,y}, w_{y,x}$ in A_ξ such that $v_{x,y}w_{x,y} = -w_{x,y}v_{x,y}$, $v_{y,x}w_{y,x} = -w_{y,x}v_{y,x}$ and $((v_{x,y}v_{y,x})^n)_{n\in\mathbb{Z}}$ is linearly independent. Such self-adjoint unitaries exist.

We define a UHF algebra $A_{[X]^2}$ by $A_{[X]^2} = \bigotimes_{\xi\in[X]^2} A_\xi \cong \bigotimes_{[X]^2\times\aleph_0} M_2(\mathbb{C})$. We define a twisted action (α, c) of the group G_X defined in Definition 5.3 on $A_{[X]^2}$ by

$$\alpha_g = \bigotimes_{x\,\in\, g \text{ and } y\,\notin\, g} \operatorname{Ad} v_{x,y}.$$

and

$$c(g,h) = \Big(\prod_{x\,\in\, g\setminus h \text{ and } y\,\in\, h\setminus g} v_{x,y}v_{y,x}\Big)\Big(\prod_{x\,\in\, g\cap h \text{ and } y\,\in\, gh} v_{x,y}v_{y,x}\Big).$$

for $g, h \in G_X$. Recall that gh is defined to be the symmetric difference between two finite subsets g and h of X. Note that the product above is finite. One can show the following.

Lemma 6.2 *The pair (α, c) is a twisted action of G_X on $A_{[X]^2}$.*

Proof. Since $v_{x,y}$ and $v_{z,t}$ commute unless $x = t$ and $y = z$, this is a routine computation. □

Let $B_{[X]^2} := A_{[X]^2} \rtimes_{(\alpha,c)} G_X$ be the twisted crossed product. In [FK1], the following is proved.

Theorem 6.3 *If X is infinite, then the C*-algebra $B_{[X]^2}$ is a direct limit of CAR algebras and it is therefore LM. It is AM if and only if $|X| > \aleph_1$.*

It is not difficult to see that $B_{[X]^2}$ is a direct limit of CAR algebras. To show that it is not AM if $|X| > \aleph_1$ we need the detailed computation and a version of Lemma 6.1. For the detail, see [FK1].

7 AM algebra which is not UHF

In this section, we give a strategy to prove that a given C*-algebra is not UHF. We need to introduce several notions.

Definition 7.1 A directed family $\{A_\lambda\}_{\lambda\in\Lambda}$ of subalgebras of a C*-algebra A is said to be *σ-complete* if for every countable directed $Z \subseteq \Lambda$ there exists $\lambda_0 \in \Lambda$ such that

$$A_{\lambda_0}\overline{\bigcup_{\lambda\in\Lambda} A_\lambda}.$$

The element $\lambda_0 \in \Lambda$ in the definition above is the supremum of Z in Λ. One can easily check the following two lemmas.

Lemma 7.2 *Every C*-algebra A has a σ-complete directed family of separable subalgebras with dense union.*

Lemma 7.3 *Let A be a C*-algebra, and let $\{A_\lambda\}_{\lambda\in\Lambda}$ be a σ-complete directed family of separable subalgebras of A with dense union. For every separable subalgebra A_0 of A there exists $\lambda \in \Lambda$ such that $A_0 \subset A_\lambda$.*

Let $A = \bigotimes_{x\in X} M_{n(x)}(\mathbb{C})$ be an infinite dimensional UHF algebra where $n\colon X \to \mathbb{Z}_{\geq 2}$ is a map from an infinite set X to the set $\mathbb{Z}_{\geq 2}$. Let $[X]^{\aleph_0}$ be the directed set of all subsets $Y \subset X$ with $|Y| = \aleph_0$ directed by the inclusion. For $Y \in [X]^{\aleph_0}$, we define $D_Y = \bigotimes_{x\in Y} M_{n(x)}(\mathbb{C}) \subset A$. Then $\{D_Y\}_{Y\in[X]^{\aleph_0}}$ is a σ-complete directed family of separable subalgebra with dense union.

Definition 7.4 For a subalgebra A_0 of a C*-algebra A, we denote by $Z_A(A_0)$ the *relative commutant* (or *centralizer*) of A_0 inside A;

$$Z_A(A_0) := \{a \in A : ab = ba \text{ for all } b \in A_0\}.$$

We avoid the common notation $A' \cap B$ for $Z_B(A)$ in order to increase the readability of certain formulas. Let $D_Y \subset A$ be as above for a UHF algebra $A = \bigotimes_{x\in X} M_{n(x)}(\mathbb{C})$. Then we have $Z_A(D_Y) = \bigotimes_{x\in X\setminus Y} M_{n(x)}(\mathbb{C})$, and hence A is generated by D_Y and $Z_A(A_\lambda)$. Now we are ready to prove the following.

Proposition 7.5 *Let A be a unital C*-algebra, and $\{A_\lambda\}_{\lambda\in\Lambda}$ be a σ-complete directed system of separable subalgebras of A with dense union. If A is a UHF algebra, then there exists $\lambda \in \Lambda$ such that A is generated by A_λ and its relative commutants $Z_A(A_\lambda)$.*

Proof. Suppose A is a UHF algebra and let $\{D_Y\}_{Y\in[X]^{\aleph_0}}$ be the σ-complete directed family of separable subalgebra with dense union defined above. Using Lemma 7.3, one can show that there exist $\lambda \in \Lambda$ and $Y \in [X]^{\aleph_0}$ such that $A_\lambda = D_Y$ in a similar way to the proof of Proposition 5.10. We are done. □

In [FK1], we prove much stronger statement than the proposition above. Namely, under weaker assumption we can prove that there are many $\lambda \in \Lambda$ such that A is generated by A_λ and its relative commutants $Z_A(A_\lambda)$. By controlling this set of λ, we can prove Theorem 3.4. For the detail see [FK2]. Using this proposition, we can now sketch the proof of Theorem 5.9.

Proof of Theorem 5.9. In Section 5, we explain the first statement. It is clear from this that B_X is UHF if $|X| = \aleph_0$. It remains to show that B_X is not UHF if $|X| > \aleph_0$. Let X be a set with $|X| > \aleph_0$. The family $\{B_Y\}_{Y\in[X]^{\aleph_0}}$ defined in the proof of Proposition 5.11 is a σ-complete directed system of separable subalgebras of B_X with dense union. One can show that if $Y \neq X$, then B_X is not generated by B_Y and its relative commutants $Z_{B_X}(B_Y)$ in a similar way to the proof of Proposition 5.10. Thus by Proposition 7.5 B_X is not UHF. □

Using a similar criterion, we construct examples of unital LM algebras which are not UHF using Jiang–Su algebra, and also examples of unital AM algebras which are not UHF using crossed products in [FK1]. These two types of examples have different properties than the ones in Theorem 5.9. In particular, the latter type shows the following which answers one question raised by Masamichi Takesaki in my talk at RIMS.

Theorem 7.6 *There exists a unital AM algebra faithfully represented on a separable Hilbert space which is not UHF.*

The following problem is still open.

Problem 7.7 *Is there an LM algebra faithfully represented on a separable Hilbert space which is not AM?*

Since $\chi(B(\ell^2(\mathbb{N}))) = 2^{\aleph_0}$, there is no such a C*-algebra by (1) of Theorem 3.1 if we assume the continuum hypothesis $2^{\aleph_0} = \aleph_1$. We do not know what happens if we do not assume the continuum hypothesis.

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